CHAPTER 4
STATISTICAL INFERENCE

Statistical inference is the process of making "inferences," or deductions, concerning large numbers.

Statistical processes are based on studies of large amounts of data. However, it is virtually impossible to examine each person or object in a large group (population). Therefore, the common practice is to select a representative sample group of manageable size for detailed study.

This may be seen in the problem of determining the average height of all 15 year old boys in the United States. It would be impossible to measure each boy; therefore, a representative group is taken from the population and measured, then an inference is made to the population.

Prior to the discussion of sampling we will review combinations, permutations, and probability distributions (which were discussed in Mathematics, Vol. 2, NavPers 10071-B) and the interpretation of standard deviation.

REVIEW

This section is the brief review of combinations, permutations, and probability. These subjects will be discussed in order that they may be related to distributions.

COMBINATIONS

A combination is defined as a possible selection of a certain number of objects taken from a group with no regard given to order. For instance, if we choose two letters from A, B, and C, we could write the letters as

AB, AC, and BC

The order in which we wrote the letters is of no concern; that is, AB could be written BA but we would still have only one combination of the letters A and B.

The general formula for possible combinations of n objects taken r at a time is

$$_nC_r = \frac{n!}{r!(n-r)!}$$

EXAMPLE: If we have available seven men and need a working party of four men, how many different groups may we possibly select?

SOLUTION: Write

$$_nC_r = \frac{n!}{r!(n-r)!}$$

where

$$n = 7$$

and

$$r = 4$$

Then,

$$_nC_r = \frac{7!}{4!(7-4)!}$$

$$= \frac{7!}{4!\,3!}$$

$$= \frac{5 \cdot 6 \cdot 7}{3 \cdot 2 \cdot 1}$$

$$= 35$$

Principle of Choice

If a selection can be made in n ways, and after this selection is made, a second selection can be made in n_2 ways, and so forth for r ways, then the r selections can be made together in

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdots n_r \text{ ways}$$

EXAMPLE: In how many ways can a coach choose first a football team and then a basketball team if twenty boys go out for either team?

SOLUTION: The coach first may choose a football team. Write

$$_nC_r = \frac{20!}{11!(20-11)!}$$

$$= \frac{20!}{11!\,9!}$$

$$= \frac{12 \cdot 13 \cdot 14 \cdot 15 \cdot 16 \cdot 17 \cdot 18 \cdot 19 \cdot 20}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 167{,}960$$

The coach then chooses a basketball team from the remaining nine boys. Write

$$_nC_r = \frac{9!}{5!(9-5)!}$$

$$= \frac{9!}{5!\,4!}$$

$$= \frac{6 \cdot 7 \cdot 8 \cdot 9}{4 \cdot 3 \cdot 2}$$

$$= 126$$

Then, by the principle of choice, the coach may choose the two teams together in

$$(167{,}960)\,(126) = 21{,}162{,}960 \text{ ways}$$

PERMUTATIONS

Permutations are similar to combinations but extend the requirements of combinations by considering order. If we choose the letters A and B, we have only one combination; that is, AB—but we have two permutations. The two permutations are AB and BA where order is considered.

The general formula for possible permutations of n objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}$$

EXAMPLE: If six persons are to fill three different positions in a company, in how many ways is it possible to fill the positions?

SOLUTION: Since any person may fill any position, we have a permutation of

$$_6P_3 = \frac{n!}{(n-r)!}$$

$$= \frac{6!}{3!}$$

$$= 4 \cdot 5 \cdot 6$$

$$= 120$$

Principle of Choice

The principle of choice holds for permutations as well as combinations.

EXAMPLE: Two positions are to be filled from a group of seven people. One position requires two people and the other requires three. In how many ways may the positions be filled?

SOLUTION: The first position may be filled by

$$_nP_r = \frac{7!}{(7-2)!}$$

$$= \frac{7!}{5!}$$

$$= 6 \cdot 7$$

$$= 42$$

and the second position may be filled using the remaining five people; that is,

$$_nP_r = \frac{5!}{(5-3)!}$$

$$= \frac{5!}{2!}$$

$$= 60$$

Therefore, both positions may be filled in

$$(42)\,(60) = 2520$$

PROBABILITY

This section covers a review of probability in a somewhat new or different approach to the subject.

For an event that will result in any of n equally likely ways, with s indicating success and f indicating failure, the probability of success is

$$p = \frac{s}{s + f}$$

where

$$s + f = n$$

EXAMPLE: What is the probability that a die will land with a six showing?

SOLUTION: There is only one successful way the die can land and there are five ways of failure, therefore,

$$s = 1$$

$$f = 5$$

and

$$p = \frac{s}{s + f}$$

$$= \frac{1}{1 + 5}$$

$$= \frac{1}{6}$$

If we assign the letter q to be the probability of failure and p the probability of success then,

$$p = \frac{s}{s + f}$$

and

$$q = \frac{f}{s + f}$$

and

$$p + q = \frac{s}{s + f} + \frac{f}{s + f} = 1$$

In the case of the die in the preceding example, the probability of failure of the six showing is

$$f = 5$$

and

$$q = \frac{f}{s + f}$$

$$= \frac{5}{1 + 5}$$

$$= \frac{5}{6}$$

and

$$p + q = \frac{1}{6} + \frac{5}{6}$$

$$= 1$$

Although the previous example is not a practical one, we will use this approach in our discussion of probability for the sake of understanding. The same rules we will discuss may be applied to practical situations, especially where the relative frequency is determined on the basis of adequate statistical samples. In these cases relative frequency is a close approximation to probability. Relative frequency is defined as the number of successful events divided by the total number of events. Relative frequency is empirical in nature; that is, it is deduced from previous occurrences.

When a coin is tossed three times, the probability that it falls heads exactly one time is shown in the outcomes as

$$\text{T T H, T H T, H T T} \qquad (1)$$

and the other outcomes are

$$\text{T T T, T H H, H T H, T H H, H H H} \qquad (2)$$

where group (1) are the favorable outcomes and group (2) are the unfavorable outcomes.

The probability that event A occurs is the ratio of the number of times A occurs to the total possible outcomes. If we let $P\{A\}$ denote the probability that event A will occur; let $n(A)$ denote the number of outcomes which produce A; and let N denote the total number of outcomes, we may show this as

$$P\{A\} = \frac{n(A)}{N}$$

$$= \frac{3}{8}$$

which is really the favorable outcomes divided by the total number of trials.

In our first example, that of tossing a die, the probability of a six showing face up, if we let B be the event of the six showing, is given by

$$P\{B\} = \frac{n(B)}{N}$$

$$= \frac{1}{6}$$

Mutually Exclusive Events

Two or more events are called mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. If two events are A and B, then

$$P\{A + B\} = \frac{n(A) + n(B)}{N}$$

$$= \frac{n(A)}{N} + \frac{n(B)}{N}$$

and

$$P\{A\} = \frac{n(A)}{N}$$

and

$$P\{B\} = \frac{n(B)}{N}$$

therefore

$$P\{A + B\} = P\{A\} + P\{B\}$$

This is called the addition rule.

EXAMPLE: What is the probability of a five or a six showing face up if a die is tossed?

SOLUTION: Let A be the five and B be the six. Then,

$$P\{A\} = \frac{1}{6}$$

and

$$P\{B\} = \frac{1}{6}$$

therefore,

$$P\{A + B\} = P\{A\} + P\{B\}$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

Dependent Events

Two or more events are said to be dependent if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others. If two events are A and B then,

$$P\{AB\} = P\{A\}\, P\{B|A\}$$

which indicates that the probability of two events occurring is equal to the probability of the first (A) times the probability of the second (B) when it is known that A has occurred. This is called the multiplication rule and $P\{B|A\}$ is referred to as conditional probability.

EXAMPLE: What is the probability of drawing, in two successive draws (one marble at a time), two black marbles if a box contains three white and two black marbles?

SOLUTION: Let the draws be A and B. Then

$$P\{AB\} = P\{A\}\, P\{B|A\}$$

and

$$P\{A\} = \frac{2}{5}$$

and

$$P\{B|A\} = \frac{1}{4}$$

therefore

$$P\{AB\} = \frac{2}{5} \cdot \frac{1}{4}$$

$$= \frac{1}{10}$$

If two events A and B do not affect each other, then A is said to be independent of B and we write

$$P\{AB\} = P\{A\}\,P\{B\}$$

where A and B are independent.

EXAMPLE: What is the probability that from a box containing three white and two black marbles we draw a white marble, replace it, and then draw a black marble?

SOLUTION: Let A and B be the events respectively, then

$$P\{AB\} = P\{A\}\,P\{B\}$$

because one event does not affect the other event. Then

$$P\{A\} = \frac{3}{5}$$

and

$$P\{B\} = \frac{2}{5}$$

therefore

$$P\{AB\} = \frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$$

DISTRIBUTIONS

In order to analyze the theory of probability through mathematical principles we must first discuss the formation of a mathematical model. While many types of data may be applied to a normal distribution curve, it should not be assumed that all sets of data conform to the curve. Data in the form of height and weight typically conform to the normal distribution curve. It is unlikely that data of a social nature would conform. Coin tossing is a form of data that does conform to the normal probability curve and we will use this type data for our discussion of distributions.

BINOMIAL

When we toss a coin three times, we list the possible outcomes as

TTT, TTH, THT, HTT, THH, HTH, HHT, HHH

Since each toss of the coin is independent and

$$P\{H\} = \frac{1}{2}$$

and

$$P\{T\} = \frac{1}{2}$$

then the probability of TTT is

$$P\{TTT\} = P\{T\}\,P\{T\}\,P\{T\} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

The probability of each of the other outcomes is the same.

If we are interested in only the number of heads obtained we let x denote this and we may have 0, 1, 2, or 3 heads. In table form this is shown as

Outcome	TTT	TTH	THT	HTT	THH	HTH	HHT	HHH
x	0	1	1	1	2	2	2	3

Now, the probability of different values of x are

$P\{0\} = \frac{1}{8}$ (TTT occurs once in the eight chances)

$P\{1\} = \frac{3}{8}$

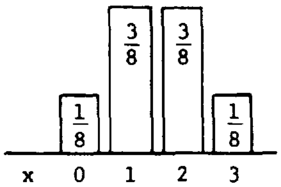
$P\{2\} = \frac{3}{8}$

$P\{3\} = \frac{1}{8}$

and in table form they appear as

x	0	1	2	3
P{x}	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

If we now put this information into a histogram the distributions will appear as



and this is a representation of a frequency distribution of the probabilities.

If we toss a die three times and let x become the number of twos showing, we may form the table, where T is the two and N is not a two, by writing

Outcome	NNN	TNN	NTN	NNT	TTN	TNT	NTT	TTT
x	0	1	1	1	2	2	2	3

Since the probabilities are different, that is

$$P\{T\} = \frac{1}{6}$$

and

$$P\{N\} = \frac{5}{6}$$

by use of the multiplication rule we may make a table as

Outcome	Probability
NNN	$\left(\frac{5}{6}\right)^3$
TNN	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
NTN	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
NNT	$\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$
TTN	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
TNT	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
NTT	$\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$
TTT	$\left(\frac{1}{6}\right)^3$

Now, since each of the groups of events are mutually exclusive—for example, where we have two twos and one non-two (TTN, TNT, and NTT)—we add these together to find P{x}. Instead of adding the same thing three times we multiply by three. This gives us the table entry for P{x} where x = 2. We complete the table as follows:

x	0	1	2	3
P{x}	$\left(\frac{5}{6}\right)^3$	$3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$	$3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$	$\left(\frac{1}{6}\right)^3$

Then

$$P\{0\} = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

$$P\{1\} = 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = \frac{75}{216}$$

$$P\{2\} = 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) = \frac{15}{216}$$

$$P\{3\} = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

If we let p be the probability of a two and q be the probability of not a two we may write

x	0	1	2	3
$P\{x\}$	q^3	$3q^2p$	$3qp^2$	p^3

This approach to frequency distributions is appropriate for small numbers of trials but when large numbers of trials are involved we rely upon the binomial distribution. This is given as

$$P\{x\} = \frac{n!}{x!(n - x)!} p^x q^{n - x}$$

Notice that

$$\frac{n!}{x!(n - x)!}$$

is really

$${}_nC_x$$

therefore

$$P\{x\} = {}_nC_x \; p^x q^{n - x}$$

In the die example the probability of one two showing is

$$P\{x\} = {}_nC_x \; p^x q^{n - x}$$

where

$$n = 3$$

$$p = \frac{1}{6}$$

$$x = 1$$

and

$$q = \frac{5}{6}$$

then

$$P\{1\} = {}_3C_1 \; p^1 q^2$$

$$= \frac{3!}{1!(2)!} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$= 3 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

which agrees with our previous answer.

EXAMPLE: What is the probability of a single one showing in three tosses of a die?

SOLUTION: Write

$$P\{x\} = {}_nC_x \; p^x q^{n - x}$$

and

$$x = 1$$

$$n = 3$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

therefore,

$$P\{1\} = {}_3C_1 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$= 3 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$= \frac{25}{72}$$

EXAMPLE: If a die is tossed five times, find the following probabilities $P\{x\}$: $x = 0, 1, 2, 3, 4, 5$.

SOLUTION: Write

$$P\{x\} = {}_nC_x \; p^x q^{n - x}$$

where

$$x = 0, 1, 2, 3, 4, 5$$

$$n = 5$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

then

$$P\{0\} = {}_5C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5$$

$$= 1(1) \left(\frac{5}{6}\right)^5$$

$$= \frac{3125}{7776}$$

$$P\{1\} = {}_5C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4$$

$$= 5 \left(\frac{1}{6}\right) \left(\frac{625}{1296}\right)$$

$$= \frac{3125}{7776}$$

$$P\{2\} = {}_5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$

$$= 10 \left(\frac{1}{36}\right) \left(\frac{125}{216}\right)$$

$$= \frac{1250}{7776}$$

$$P\{3\} = {}_5C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2$$

$$= 10 \left(\frac{1}{216}\right) \left(\frac{25}{36}\right)$$

$$= \frac{250}{7776}$$

$$P\{4\} = {}_5C_4 \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1$$

$$= 5 \left(\frac{1}{1296}\right) \left(\frac{5}{6}\right)$$

$$= \frac{25}{7776}$$

$$P\{5\} = {}_5C_5 \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0$$

$$= 1 \left(\frac{1}{7776}\right) (1)$$

$$= \frac{1}{7776}$$

EXAMPLE: Find, in the preceding problem, the probability of at least four threes showing, that is

$$P\{x \geq 4\}$$

SOLUTION: We desire the probability of both $P\{4\}$ and $P\{5\}$. These are mutually exclusive, therefore we use the addition rule and write

$$P\{x \geq 4\} = P\{4\} + P\{5\}$$

$$= \frac{25}{7776} + \frac{1}{7776}$$

$$= \frac{26}{7776}$$

Notice that if we add all the probabilities together, we find the probability that any of the events will happen, which is,

$$P\{0\} + P\{1\} + P\{2\} + P\{3\} + P\{4\} + P\{5\}$$

$$= \frac{3125}{7776} + \frac{3125}{7776} + \frac{1250}{7776} + \frac{250}{7776} + \frac{25}{7776} + \frac{1}{7776}$$

$$= \frac{7776}{7776} = 1$$

This corresponds with our previous assumption that the sum of successful and failing events equals 1.

EXAMPLE: If a die is tossed eight times, what is the probability that a four will show exactly twice?

SOLUTION: Write

$$P\{x\} = {}_nC_x\, p^x\, q^{n-x}$$

where

$$n = 8$$

$$x = 2$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

then

$$P\{2\} = {}_8C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

$$= 28 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^6$$

EXAMPLE: If a die is tossed four times, what is the probability that a three will show at "most" two times?

SOLUTION: We must find the sum of $P\{0\}$, $P\{1\}$, and $P\{2\}$ therefore we write

$$P\{x\} = {}_nC_x\, p^x\, q^{n-x}$$

where

$$n = 4$$

$$x = 0, 1, 2$$

$$p = \frac{1}{6}$$

$$q = \frac{5}{6}$$

and

$$P\{0\} = {}_4C_0 \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4$$

$$= 1\ (1) \left(\frac{625}{1296}\right)$$

$$= \frac{625}{1296}$$

$$P\{1\} = {}_4C_1 \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3$$

$$= 4 \left(\frac{1}{6}\right) \left(\frac{125}{216}\right)$$

$$= \frac{500}{1296}$$

$$P\{2\} = {}_4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2$$

$$= 6 \left(\frac{1}{36}\right) \left(\frac{25}{36}\right)$$

$$= \frac{150}{1296}$$

then

$$P\{x \le 2\} = P\{0\} + P\{1\} + P\{2\}$$

$$= \frac{625}{1296} + \frac{500}{1296} + \frac{150}{1296}$$

$$= \frac{1275}{1296}$$

In problems of a binomial nature, four properties are required. They are as follows:

1. The number of trials must be fixed.

2. Each trial must result in either a success or a failure.

3. The probability of successes must be identified.

4. All of the trials must be independent.

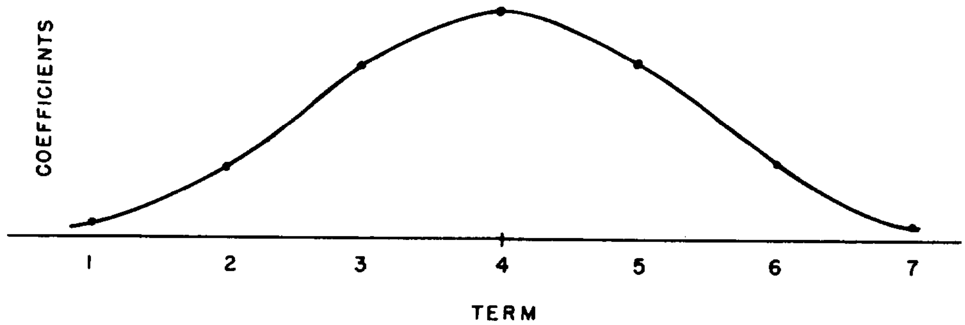


Figure 4-1.—Graph of coefficients of $(x + y)^n$, n = 6

In the previous problems we have made use of the formula

$$P\{x\} = {}_nC_x \, p^x \, q^{n-x}$$

When we plot the coefficient of p and q, that is,

$${}_nC_x$$

we find that the curve will resemble that shown in figure 4-1, depending on the value set for n.

NORMAL

When information from a large population is examined it will be found that there will be many deviations from the mean. Both positive and negative deviations will occur with nearly the same frequency. Also small deviations will occur more frequently than large deviations.

Many years ago an equation was determined by De Moivre and later it was applied more broadly to areas of measurement by Laplace and Gauss. This equation which exhibits the previous mentioned characteristics of a population, is

$$y = ke^{-h^2x^2}$$

We will treat h and k as constants of one. The equation then becomes

$$y = e^{-x^2}$$

where e is the base of the system of natural logarithms and equals approximately $\frac{11}{4}$. The equation, then, may be written as

$$y = \left(\frac{4}{11}\right)^{x^2}$$

When we assign values to x and derive the values for y a curve is developed as shown in figure 4-2.

It has been found, when the entire area under the curve equals one, that

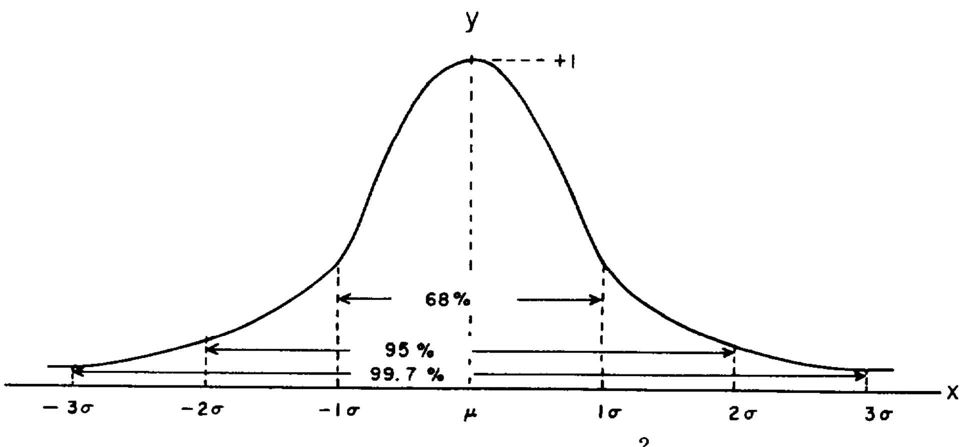


Figure 4-2.—Graph of $y = e^{-x^2}$

$$\mu \pm 1\sigma = 68 \text{ percent of the area}$$

$$\mu \pm 2\sigma = 95 \text{ percent of the area}$$

and

$$\mu \pm 3\sigma = 99.7 \text{ percent of the area}$$

(These areas are shown in figure 4-2.)

These areas also represent probabilities that a single variable will fall within these intervals. To use the table of areas under the normal curve the transformation formula

$$z = \frac{x - \mu}{\sigma}$$

must be applied.

We call the value z the standard normal deviate. This indicates the number of standard deviations the variable x is above or below the mean.

EXAMPLE: Find the area under the normal curve from z equal zero to z equal 1.5.

(This area is shown in figure 4-3.)

SOLUTION: In table 4-1 read down the first column to 1.5 then across to 0.00 and find 0.4332.

EXAMPLE: Find the area under the normal curve from z equal -0.4 to z equal 0.5.

(This area is shown in figure 4-4.)

SOLUTION: The table gives only areas from z equal zero to some positive value, therefore we rely on symmetry to find the area from z equal zero to z equal -0.4. Find the area from

$$z = 0$$

to

$$z = 0.4$$

as

$$0.1554$$

Then, the area from

$$z = 0$$

to

$$z = 0.5$$

as

$$0.1915$$

We now add the area

$$\begin{array}{rl} 0.1554 & \\ +\ 0.1915 & \\ \hline 0.3469 & \text{total area desired} \end{array}$$

EXAMPLE: If we are given the distribution as shown in figure 4-5, what is the area between x equal 110 and x equal 125, if the mean is 120 and the deviation is 5? (We assume normal distribution.)

SOLUTION: We use

$$z = \frac{x - \mu}{\sigma}$$

to find the values of z which correspond to

$$x = 110$$

and

$$x = 125$$

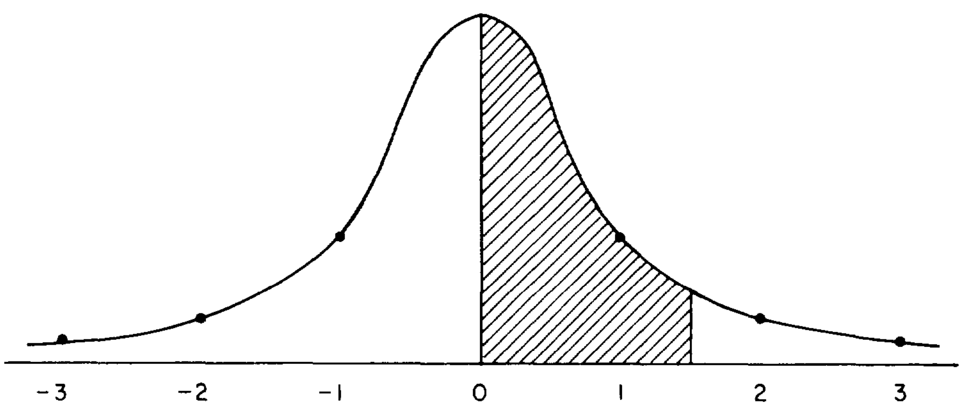


Figure 4-3.—Normal curve (z = 1.5).

Table 4-1.--Areas under the normal curve.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.5000
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000

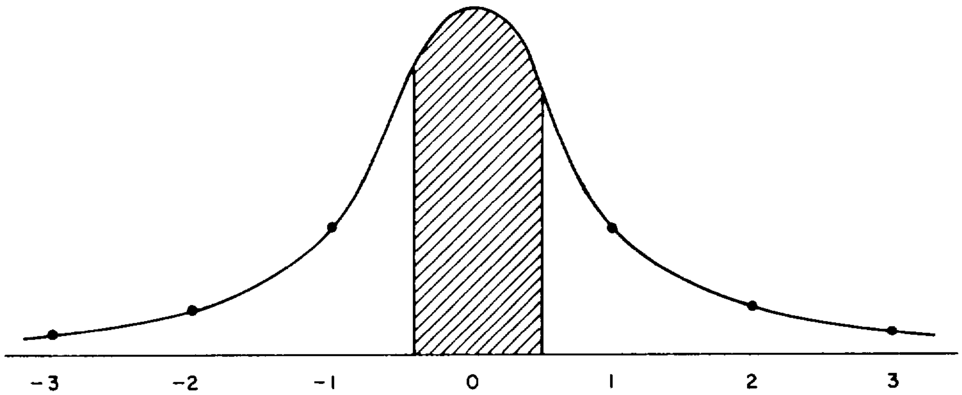


Figure 4-4.—Normal curve (z = 0.5).

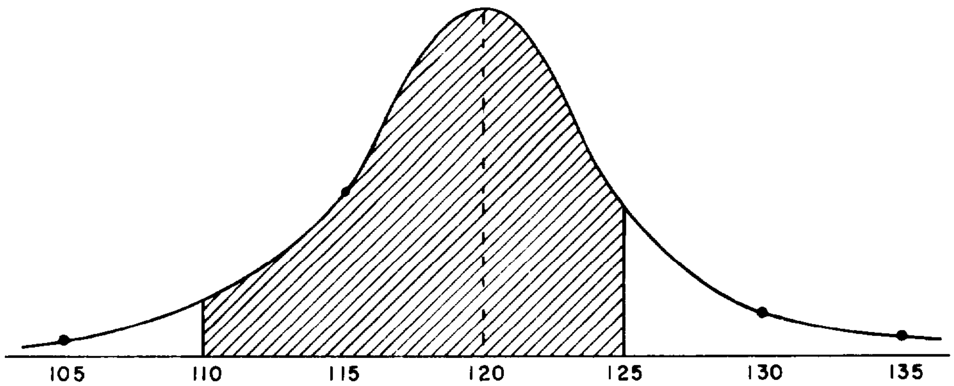


Figure 4-5.—Normal curve (x = 110 to 125).

We write

$$z = \frac{110 - 120}{5}$$

$$= -\frac{10}{5}$$

$$= -2$$

which corresponds to x equal 110. Then,

$$z = \frac{125 - 120}{5}$$

$$= \frac{5}{5}$$

$$= 1$$

which corresponds to x equal 125. By use of table 4-1 find the area from z equal zero to -2 to be 0.4772 and the area from z equal zero to +1 to be 0.3413. We then add the areas and find

$$\begin{array}{r} 0.4772 \\ +\ 0.3413 \\ \hline 0.8185 \end{array} \text{ total area desired}$$

While we have discussed areas under the curve, these areas are also the probabilities of occurrence; that is, in the preceding example, the probability that a single selected value will fall between 110 and 125 is 0.8185. These probabilities hold regardless of the value of the mean or standard deviation as long as the values form a normal distribution.

In problems of the previous type, it is advisable to draw a rough sketch of the curve to picture the areas or probabilities desired.

EXAMPLE: With a set of grades which form a normal distribution and have a mean of 70 and a standard deviation of 6, what is the probability that a grade selected at random will be higher than 78?

SOLUTION: Sketch the curve as shown in figure 4-6. The area or probability we desire is shaded. We find this probability by finding the probability of the grades above 70 then subtracting the probability of the grades from 70 to 78. We write

$$z = \frac{78 - 70}{6}$$

$$= \frac{8}{6}$$

$$= 1.33$$

In table 4-1

$$z = 1.33$$

is

$$0.4082$$

and the probability of grades above 70 is

$$0.5000$$

then

0.5000 probability of grades above 70

- 0.4082 probability of grades from 70 to 78

0.0918 probability of grades above 78

Therefore, the probability that the grades selected will be higher than 78 is 0.0918.

POISSON

When we are faced with problems which have more outcomes than those of a binomial nature, that is 0 or 1, yes or no, or true or false, the Poisson distribution may be used. This distribution is defined with respect to a unit of measure where there may be several outcomes within the given unit of measure. It is used when the number of trials is extremely large and the probability of success in any trial is quite small.

The Poisson distribution is useful in quality control. An example may better explain this idea.

If we were to inspect boxes of resistors on a production line we might find 0, 1, 2, or more defective resistors per box. We could count the defective resistors but it would be impractical to count the number of good resistors. In this case we could use the Poisson distribution to find the probability of 0, 1, 2, or more defective resistors in a given box.

The formula for the Poisson probability is

$$P\{x\} = \frac{e^{-m}\, m^x}{x!} \quad \text{where } x = 0, 1, 2, 3, \ldots$$

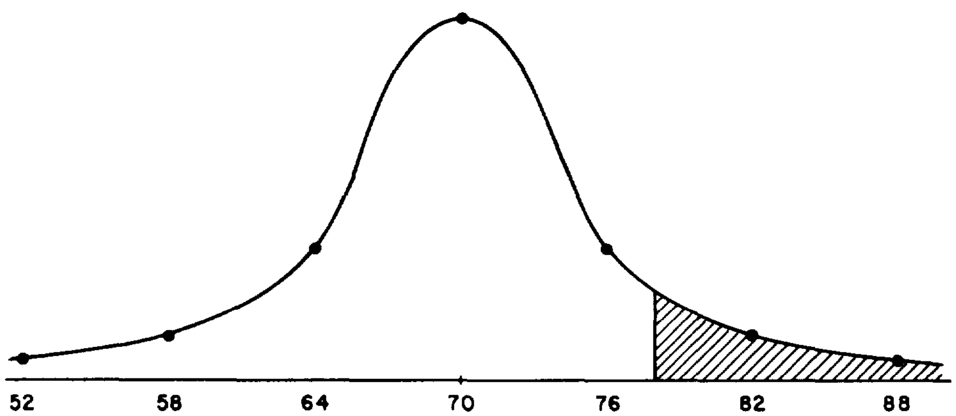


Figure 4-6.—Normal curve ($x \geq 78$).

x is the number of occurrences per unit, m is the average or mean of the occurrences per unit, and e is the base of the natural logarithms.

One requirement of the Poisson formula of probability is that the number of possible occurrences in any unit is large while the probability of a particular occurrence is small. A second requirement is that the particular occurrences in one unit do not influence the particular occurrences in another unit. Finally, the third requirement is that the average or mean must remain constant.

EXAMPLE: In our example problem suppose that on the average there were 2 defective resistors per box. What is the probability that there will be no defective resistors in a given box?

SOLUTION: Write

$$P\{x\} = \frac{e^{-m} m^x}{x!}$$

then

$$x = 0$$

$$e = 2.7$$

$$m = 2$$

therefore

$$P\{0\} = \frac{2.7^{-2}\ 2^0}{0!}$$

$$= \frac{2^0}{2.7^2}$$

$$= \frac{1}{7.29}$$

$$= 0.137$$

Notice that only the average or mean is a parameter of the Poisson distribution.

To continue our problem further, the probability of a random selected box having 3 defective resistors is

$$P\{3\} = \frac{e^{-2}\ 2^3}{3!}$$

$$= \frac{2.7^{-2}\ 8}{6}$$

$$= \frac{8}{7.29\ (6)}$$

$$= \frac{8}{43.74}$$

$$= 0.18$$

In the example given we have used the formula to determine the probabilities. Tables of the Poisson probabilities have been determined to alleviate this laborious process.

EXAMPLE: If on the average one person entered a store every 5 seconds and persons entered at random, what is the probability that in a selected 5-second period of time 3 people enter the store?

SOLUTION: Write

$$P\{x\} = \frac{e^{-m} m^x}{x!}$$

and

$$e = 2.7$$

$$x = 3$$

$$m = 1$$

therefore

$$P\{3\} = \frac{e^{-1}\ 1^3}{3!}$$

$$= \frac{2.7^{-1}\ (1)}{6}$$

$$= \frac{1}{2.7(6)}$$

$$= 0.06$$

Rather than solve this problem by use of the formula we could have used table 4-2. The table is formed with the value of x versus the value of m. In this problem we would have followed x equal 3 down until it fell in the row where m equals 1 which gives 0.061.

EXAMPLE: If aircraft arrive randomly at a field on the average of two every fifteen minutes, what is the probability that six aircraft will arrive in a given quarter-hour?

Table 4-2.--Poisson distribution (partial table).

$$f(x) = \frac{e^{-m}m^x}{x!}$$

m \ x	0	1	2	3	4	5	6	7	8	9
0.10	905	090	005							
0.15	861	129	010							
0.20	819	164	016	001						
0.25	779	195	024	002						
0.30	741	222	033	003						
0.40	670	268	054	007	001					
0.50	607	303	076	013	002					
0.60	549	329	099	020	003					
0.70	497	348	122	028	005	001				
0.80	449	359	144	038	008	001				
0.90	407	366	165	049	011	002				
1.00	368	368	184	061	015	003	001			
1.10	333	366	201	074	020	004	001			
1.20	301	361	217	087	026	006	001			
1.30	273	354	230	100	032	008	002			
1.40	247	345	242	113	039	011	003	001		
1.50	223	335	251	126	047	014	004	001		
1.60	202	323	258	138	055	018	005	001		
1.70	183	311	265	150	063	022	006	001		
1.80	165	298	268	161	072	026	008	002		
1.90	150	284	270	171	081	031	010	003	001	
2.00	135	271	271	180	090	036	012	003	001	
2.10	122	257	270	189	099	042	015	004	001	
2.20	111	244	268	197	108	048	017	005	002	
2.30	100	231	265	203	117	054	021	007	002	
2.40	091	218	261	209	125	060	024	008	002	001
2.50	082	205	257	214	134	067	028	010	003	001
2.60	074	193	251	218	141	074	032	012	004	001

SOLUTION: Write

$$P\{x\} = \frac{e^{-m}\, m^x}{x!}$$

where

$$x = 6$$

$$m = 2$$

then, by use of table 4-2 find that

$$P\{6\} = 0.012$$

EXAMPLE: A certain type aircraft averages 1.1 failures per squadron requiring ground maintenance for every 24 hour day. What is the probability of a squadron having 4 or more aircraft grounded for maintenance on a particular day?

SOLUTION: We must find the probabilities of x equal 4, 5, 6, ... By use of table 4-2 we find

$$P\{4\} = 0.020$$

$$P\{5\} = 0.004$$

$$P\{6\} = 0.001$$

$$P\{7\} = 0.000$$

therefore,

$$P\{x \geq 4\} = 0.020 + 0.004 + 0.001$$

$$= 0.025$$

Poisson to Binomial Approximation

The Poisson approximation to the binomial holds if the value of n is large and the value of p is small. Generally, if the ratio of n to p is near 1000 we can use the Poisson to approximate the binomial; that is, if n is greater than 10 and p is less than 0.01 the ratio of n/p is 1000.

In order to approximate a binomial we set np equal to m and use the Poisson table. That is, if we are sampling 50 items which have a probability of defect of 0.05 on the average, we write

$$n = 50$$

$$p = 0.05$$

and

$$np = m$$

$$(50)(0.05) = 2.5$$

then we may estimate the probability of a number of defects by using table 4-2. In this case

$$P\{0\} = 0.082$$

$$P\{1\} = 0.205$$

$$P\{2\} = 0.257$$

Normal to Binomial Approximation

When n was large and p was small, the Poisson distribution could be used to approximate the binomial. When n is large and p is neither small nor large (not close to 1 or 0), we can use the normal to approximate the binomial. To use this approximation the product of np should be equal to or greater than 5; that is, if n were 20 then

$$np \geq 5$$

$$20p \geq 5$$

$$p \geq \frac{5}{20}$$

$$\geq \frac{1}{4}$$

$$\geq 0.25$$

With n equal to 20, p should be 0.25 or greater in order that the distribution be nearly normal. Steps in using the approximation are:

1. Let np equal μ.
2. Let $\sqrt{npq}$ equal σ.
3. If finding the probabilities of the same number or less successes add 0.5 to x and if finding more successes subtract 0.5 from x (this is due to the binomial being a discrete distribution).
4. Use the normal table.

EXAMPLE: If the probability of a defective item is 0.2 and we use a sample of 500 items from a large population, what is the probability of 120 or more defective items?

SOLUTION: Write

$$\mu = np$$

$$= 500(0.2)$$

$$= 100$$

and

$$\sigma = \sqrt{npq}$$

$$= \sqrt{500\ (0.2)\ (0.8)}$$

$$= \sqrt{80}$$

$$= 8.9$$

Then,

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{119.5 - 100}{8.9}$$

$$= 2.19$$

Using table 4-1 find that the probability

$$P\{z > 2.19\} = 0.5000 - 0.4857$$

$$= 0.0143$$

In this same example, what is the probability of exactly 120 defective items? Since the probability of more than 120 defective items in the binomial distribution is the same as 120.5 defective items in the normal distribution we use x equal 120.5. (Again, this is due to the binomial being a discrete distribution.) Then

$$\mu = 100$$

$$x = 120.5$$

$$\sigma = 8.9$$

and we write

$$z = \frac{120.5 - 100}{8.9}$$

$$= \frac{20.5}{8.9}$$

$$= 2.3$$

Using table 4-1 find that

$$P\{z > 2.3\} = 0.5000 - 0.4893$$

$$= 0.0107$$

The probability of exactly 120 defective items is

$$P\{2.19 < z < 2.3\} = 0.0143 - 0.0107$$

$$= 0.0036$$

The preceding example is illustrated in figure 4-7.

EXAMPLE: If the probability of success in a single try is $\frac{1}{3}$, what is the probability of at least 6 successes in 15 tries?

SOLUTION: Write

$$n = 15$$

$$p = \frac{1}{3}$$

$$q = \frac{2}{3}$$

and

$$\mu = np$$

$$= 15\left(\frac{1}{3}\right)$$

$$= 5$$

and

$$\sigma = \sqrt{npq}$$

$$= \sqrt{15 \cdot \frac{1}{3} \cdot \frac{2}{3}}$$

$$= \sqrt{\frac{30}{9}}$$

$$= 1.825$$

then

$$z = \frac{x - \mu}{\sigma}$$

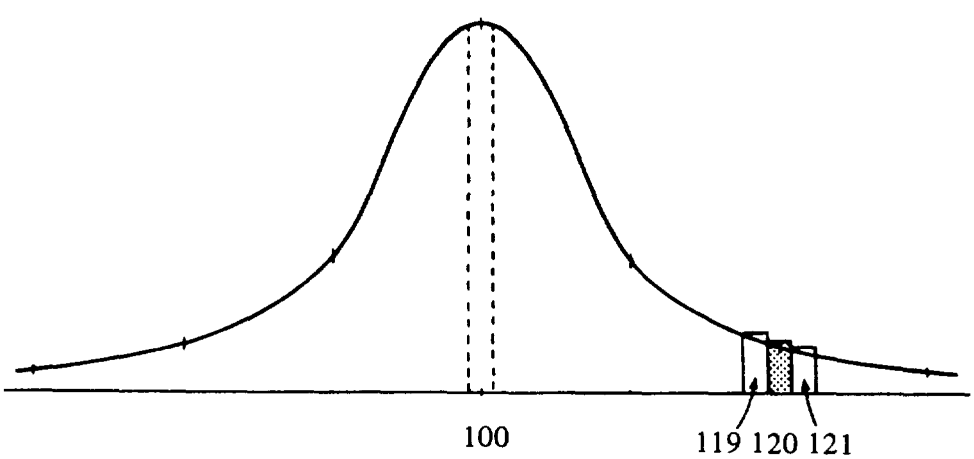


Figure 4-7.—Normal curve (x = 120.5).

(where 0.5 is subtracted from x)

$$z = \frac{6 - 0.5 - 5}{1.825}$$
$$= \frac{5.5 - 5}{1.825}$$
$$= \frac{.5}{1.825}$$
$$= 0.27$$

and by use of table 4-1 find that

$$P\{z > 0.27\} = 0.5000 - 0.1064$$
$$= 0.3936$$

INTERPRETATION OF STANDARD DEVIATION

We have discussed distribution and how special cases may approximate the normal distribution. When a normal distribution is determined the standard deviation (σ) enables us to determine characteristics of the distribution. It has been found that 99.7 percent of all items of a normal distribution fall within three standard deviations of the mean.

In table 4-3 the f column is the fractional part of the standard deviation and the A column is the area under the normal curve for ± the fractional part indicated. That is, the area under the normal curve which falls within $\pm 1.3\ \sigma$ is 0.807 or 80.7 percent.

Table 4-3.—Values for ± standard deviations.

f	A	f	A
0.1	0.080	1.6	0.891
0.2	0.159	1.7	0.911
0.3	0.236	1.8	0.928
0.4	0.311	1.9	0.943
0.5	0.383	2.0	0.955
0.6	0.451	2.1	0.964
0.7	0.516	2.2	0.972
0.8	0.576	2.3	0.979
0.9	0.632	2.4	0.984
1.0	0.683	2.5	0.988
1.1	0.729	2.6	0.991
1.2	0.770	2.7	0.993
1.3	0.807	2.8	0.995
1.4	0.838	2.9	0.996
1.5	0.866	3.0	0.997

In a set of data which is normally distributed and has a mean of 78 and a standard deviation of 6; that is,

$$\bar{x} = 78$$

and

$$\sigma = 6$$

the area under the curve between $+0.5\ \sigma$ and $-0.5\ \sigma$ is 0.383. This means that 38.3 percent of the data will fall within the range of

$$\bar{x} + 0.5\ \sigma \text{ and } \bar{x} - 0.5\ \sigma$$

or within

$$78 + 0.5(6) \text{ and } 78 - 0.5(6)$$

which is

$$81 \text{ and } 75$$

STANDARD SCORES

In many cases it is necessary to combine scores or achievement ratings *from different* tests into a single grade or rating. To combine raw scores from different tests is not statistically sound unless the means and standard deviations of the different tests are the same. The probability of this occurring is quite small; therefore, we resort to the standard score.

When we change raw scores into standard scores, we assume the raw scores form a normal distribution. The standard scores are expressed in standard deviations with a mean of zero and a standard deviation of one.

Standard scores may be added and averaged with equal weight given to each different score. As an example, we may desire an overall merit rating of persons who were given several tests. While some would score high in a particular field they may score low in others and furthermore, the difficulty of one test may vary from the difficulty of another. What we desire is a rating for each area tested in a form that can be compared with ratings of other areas tested. For this we use the standard score.

The standard score, for a particular test, is determined by the formula

$$\text{Standard Score} = \frac{\text{Raw Score - Mean}}{\text{Standard Deviation}}$$

$$= \frac{x - \bar{x}}{s}$$

EXAMPLE: If the raw scores on an examination were 68, 70, 73, 76, 81, 90, and 95, convert these to standard scores.

SOLUTION: We must find $\bar{x}$ and s. Write

$$\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{553}{7}$$

$$= 79$$

and

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

$$= \sqrt{\frac{1}{6}(628)}$$

$$= \sqrt{104.66}$$

$$\simeq 10.23$$

Then write

x	$x - \bar{x}$	$\frac{x - \bar{x}}{s}$	Standard Score
95	16	16 ÷ 10.23	1.56
90	11	11 ÷ 10.23	1.07
81	2	2 ÷ 10.23	0.19
76	-3	-3 ÷ 10.23	-0.29
73	-6	-6 ÷ 10.23	-0.58
70	-9	-9 ÷ 10.23	-0.87
68	-11	-11 ÷ 10.23	-1.07

The positive standard scores indicate that factor of standard deviations above the mean of the raw scores and negative scores indicate that factor of standard deviations below the mean of the raw scores.

If a person, on five different tests, had standard scores of +1.32, -0.93, +2.03, +0.20, and -1.20, his average standard score would be

$$\begin{array}{r} +1.32 \\ -0.93 \\ +2.03 \\ +0.20 \\ -1.20 \\ \hline +1.43 \div 5 \end{array}$$

or

$$\frac{+1.43}{5} = +0.28$$

which indicates an achievement of 0.28 standard deviations above the mean.

When standard scores with a standard deviation of one and a mean of zero are determined they involve the use of positive and negative decimals. To eliminate the use of negative scores and decimals a linear transformation may be made by the use of a greater mean and a greater standard deviation.

An example of this type transformation is made on the Graduate Record Examination. The scores on the G.R.E. are expressed using a standard deviation of 100 and a mean of 500.

To change a standard score to a corrected standard score with a mean of 500 and a standard deviation of 100 write

$$\text{Standard Score} = \frac{x - \bar{x}}{s}(s_c) + \bar{x}_c$$

where

s_c = new standard deviation (100 in our case)

and

$\bar{x}_c$ = new mean (500 in our case)

Therefore, if the standard score is 0.6 then the corrected standard score is

$$0.6\,(\sigma_c) + \bar{x}_c$$

$$= 0.6\,(100) + 500$$

$$= 60 + 500$$

$$= 560$$

EXAMPLE: Change the standard score of -1.3 to a distribution with a mean of 500 and a standard deviation of 100.

SOLUTION: Write

$$\text{Standard Score (S.S.)} = -1.3$$

$$\sigma_c = 100$$

$$\bar{x}_c = 500$$

then the corrected standard score is

$$-1.3\,(100) + 500$$

$$= -130 + 500$$

$$= 370$$

PROBLEMS: Find the corrected standard score in the distribution indicated of the given standard scores.

1. 1.3 in distribution with $\bar{x}_c = 50$ and $\sigma_c = 10$.

2. -2.4 in distribution with $\bar{x}_c = 50$ and $\sigma_c = 10$.

3. -0.3 in distribution with $\bar{x}_c = 100$ and $\sigma_c = 20$.

ANSWERS:

1. 63

2. 26

3. 94

SAMPLING

When sampling a population one should be careful not to allow any preventable bias from becoming part of the sample data.

If a sample of the population were being taken to determine the average height of 16-year-old boys a bias would be introduced if the sample came from basketball players because they would probably be the tallest boys in the population. Bias in a sample will cause the predicted results to be in error.

A classic example of bias in a sample was encountered in the Literary Digest poll conducted in 1936 which resulted in the prediction that Landon would be elected President. Roosevelt was elected and upon investigation of the methods of sampling it was found that the sample was taken from persons who had a telephone and from those who had an automobile. This resulted in only a certain income group being polled which introduced a bias.

In order to select a random sample which will have little bias, the conscious selection of sample must not enter into the selection process. The use of a table of random numbers will aid in the removal of bias from the sample.

One use of a sample is to make a prediction about the population. It is known that the means of samples follow the normal distribution even though the population may vary somewhat from a normal distribution. Our predictions, from a sample, about the population are made with a certain level of confidence. The standard error of the mean allows us to give a level of confidence concerning the sample mean.

The standard error of the mean is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

where N is the number in the sample and σ is the standard deviation. Since the sample means follow the normal distribution curve we may use table 4-1. Note that N should be greater than 25.

We have previously determined that $\bar{x} \pm 1\sigma$ contains about 68 percent of the items. We determine this from table 4-1 by finding 1.0 and reading .3413. Our table gives only the positive values above the mean; therefore, we double this figure (because of symmetry) and find

$$\begin{array}{l} .3413 \\ \underline{.3413} \\ .6826 \text{ or 68 percent} \end{array}$$

When we speak of some level of confidence, such as 5 percent, we mean there is less than a 5 percent chance that the sample mean will differ from the population mean by some given amount.

We will use an example to illustrate the preceding statements. If we desired to find the average height of 16-year-old boys in a certain city we could select at random 36 boys and measure their heights. From the data collected suppose we find that $\bar{x} = 66$ inches and $\sigma = 2$ inches. Our problem now is to estimate the mean and standard deviation of the population. The standard error of the mean is

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

$$= \frac{2}{\sqrt{36}}$$

$$= \pm \frac{2}{6}$$

$$= \pm\ 0.3$$

We may now state that there is a .68 probability that the population mean is within the range of

$$\bar{x} \pm 1\sigma_{\bar{x}}$$

or

$$66 \pm (1)(.3)$$

or

$$66.3 \text{ and } 65.7$$

Our level of confidence is 32 percent which means that there is less than 32 percent chance that the sample mean differs from the true mean by $\bar{x} \pm 1\sigma_x$.

If we increase the size of the sample we will obtain a smaller range for our confidence level of 32 percent. Suppose we increase our sample from 36 to 100. Then

$$\bar{x} = 66$$

$$\sigma = 2$$

and

$$\sigma_{\bar{x}} = \frac{2}{\sqrt{100}}$$

$$= \pm \frac{2}{10}$$

$$= 0.2$$

and we now have a .68 probability that the population mean is within the range of

$$\bar{x} \pm 1\sigma_{\bar{x}}$$

or

$$66 \pm (1)(.2)$$

or

$$66.2 \text{ and } 65.8$$

as compared to

$$66.3 \text{ and } 65.7$$

when our sample number was 36.

In the preceding discussion it must be understood that the population should be large compared to the sample size.

EXAMPLE: Given a sample where $\bar{x} = 70$ and $\sigma = 4$ and $N = 100$ find the range about the mean which gives a 5 percent confidence level.

SOLUTION: We want a .95 probability that the true mean will be within a range to be determined.

Write

$$\bar{x} = 70$$

$$\sigma = 4$$

$$N = 100$$

and

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

$$= \frac{4}{\sqrt{100}}$$

$$= \pm \frac{4}{10}$$

$$= \pm\ .4$$

We desire .95 probability; therefore, we divide by 2 because our table values are for one-half the total area.

$$\frac{.95}{2} = .475$$

and find in table 4-1 that .475 is given for a factor of $\sigma_{\overline{x}}$ of 1.96.
Then, the range we desire is

$$\overline{x} \pm 1.96\ \sigma_{\overline{x}}$$

or

$$70 \pm 1.96\ (.4)$$

which is

$$70 \pm .784$$

or

$$70.784 \text{ and } 69.216$$

Thus, the probability is .95 that the true mean lies in the range

$$70.784 \text{ and } 69.216$$

PROBLEM: A random sample of forty resistors from an extremely large supply reveals a mean resistance of 1000 ohms and a standard deviation of 5 ohms. Find the standard error of the mean. What is the range about the mean of the sample which will give a .90 probability that the true resistance value of all the resistors will fall within it?

ANSWER:

(a) $\sigma_{\overline{x}} = .8$

(b) 1000 ± 1.32 or 1001.32 and 998.68